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Abstract

Competitive screening and information transmission

by Inácio Bó and Chiu Yu Ko*

We consider a simple model of the competitive screening of students by schools and colleges. Students apply to schools which then perform costly screening procedures of the applicants to select those with high ability. Students who receive more than one offer may choose among those. Colleges select students and can observe the school which they attended. We show a channel through which students' preferences affect schools' screening decisions and outcomes: as schools increase the screening for high-ability students, a greater proportion of them is identified as such by multiple schools and are able to select one among them to attend. Schools' marginal gains from screening therefore depend on other schools' screenings and students' preferences. By focusing on the schools' screening choices (instead of the students' application decisions), we show how the competition for students between schools and colleges affect outcomes and students' welfare. We also show that, simply by observing which school a candidate attended, colleges can "free-ride" on the information produced by a fierce competition between schools for those students. Finally, we show that although colleges make full use of the information contained in the school a student attended, the extent to which students can improve the college that they are matched to by going to a (less desired) high-ranked school is fairly limited.

Keywords: Information transmission, college admissions, screening, rankings

JEL classification: C78, D61, D83

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1 Introduction

Colleges, and in many cases schools and high schools, often face a number of applicants larger than their own capacities. At the same time, many of these institutions value the composition of the cohort of students admitted in terms of their abilities in the relevant fields. In other words, it is often the case that colleges and schools want to admit the "best and the brightest" among their applicants. To identify those high-ability students from the pool of applicants, however, involves engaging in costly examinations, interviews, and other applicant screening methods. As students move through the education system, the institutions where the student comes from can be used as a signal of the student's ability. For example, if high schools also engage in screening applicants when selecting them, when a college sees that an applicant graduated from a very competitive high school, this fact may carry information about the likelihood of a student having high ability.

This paper evaluates the effect that these screening strategies have on the matching of students to institutions, when we take into account three aspects of this process. The first is that screening decisions affect other institutions' outcomes, since the same student may be identified as high ability by more than one institution. The second is that students' preferences affect the returns from screening, since a student will only accept an offer from a less preferred institution if she did not receive an offer from a more preferred one. The third is how the information generated by this knowledge about screening and preferences is used by institutions at the next level in the educational sequence.

By incorporating these elements into a simple model, we provide insights into multiple aspects of the process in closed-form solutions. In our model, there are two schools and two colleges. Students can apply to the schools or colleges at no cost. This abstracts the strategic aspect of the choice of where to apply to and, as a result, isolates the effect of the intersection of schools' offers on outcomes and schools' incentives. When a school engages in screening and identifies a high-ability student, the other school may also identify that same student as such. The student will then have to choose between the offers made based on her preference over these schools. Therefore, the less popular school will see a greater proportion of its offers rejected than the more popular school. That translates into a higher cost per high-ability student admitted, or equivalently, a lower marginal gain from screening, for the former.

These different marginal returns from screening result in a surprising relation between screening costs and the schools' success in obtaining high-ability students. We show that, while in general a reduction in screening cost improves outcomes for both schools, there is a point in which further reductions in cost have opposite effects on the student cohorts obtained by them: the popular school accepts more high-ability students, while the less popular one obtains less of them. The reason is that lower screening costs increase the incentive for schools to engage in it and, as a result, it also increases the number of students with offers from both popular and unpopular schools. Since unpopular schools are negatively affected by an increase in the number of offers that students have, when costs are low enough that effect becomes stronger than the reduction in the marginal cost of screening (Proposition 1).

Another interesting observation is how changes in students' preferences affect schools' strategies and, as a result, their own welfare. As students' preferences become less correlated (that is, the less popular school increases in popularity), this creates two opposite effects in both schools. The return from screening for the popular school decreases and the return from screening for the less popular school increases, due to the effect created by simultaneous offers. We show, however, that the former effect is proportional to the amount of screening of the more popular school, whereas the latter is proportional to the amount of screening of the less popular school. As a result, there is an overall increase in the equilibrium amount of screening performed by the schools, leading also to an increase in the number of students who receive multiple offers (Proposition 6). While this increases the total number of high-ability students matched to the schools, it paradoxically reduces the number of students who are matched to their most preferred school (Proposition 7). This happens mainly because when a student changes her preference toward the less popular school, she will prefer to be matched to a school that screens fewer students. As a result, the likelihood that she will receive an offer from her top school is lower.

Our model also explores the transmission of information that takes place when colleges make screening decisions knowing which schools the students come from. The principle is simple: since in equilibrium the proportion of high-ability students is higher at the most popular school, colleges can explore that fact when screening students. Moreover, that ability to explore this information should also depend on students' preferences over colleges: if students have common preferences over colleges, the most preferred college can make its decisions independently (since all its offers will be accepted). Therefore, whatever informational advantage there is to be exploited, it will be fully available for the top college. We show, however, that this does not translate into an advantage on the part of the top college in exploiting that information: the number of high-ability students obtained by both colleges depends on the total number of those in the schools, but not on how they are distributed among them. The colleges do enjoy the benefits that are generated by the schools' reaction to changes in preferences, though: colleges obtain better cohorts, in equilibrium, when students' preferences over schools are less homogeneous. This happens because, in that case, colleges "free ride" on the increased screening performed by the schools (Proposition 8). Students who went to the most popular school, however, have better chances of being accepted at the top college.

Finally, we explore the question of whether students would, in equilibrium, strategically choose which school to attend in order to obtain better outcomes at the college level. We show that, at least when cardinal preferences are common among students who have the same ordinal preferences, there is no incentive to do so with the objective of obtaining better chances at the top college, since at the optimal this college will screen the same proportion of high-ability students from each school. When the top college fills its capacity with high-ability students and the less popular college does not, however, students may reduce their probability of remaining unmatched by strategically choosing which school to attend.

1.1 Related Literature

Our paper is closely related to recent papers that focus on information acquisition in matching problems. There is a line of research that analyzes the conditions under which assortative matchings are produced. While previous work showed that search costs in the form of time discounting may lead to non-assortative matchings in a market with transfers, Atakan [2006] showed that this is not the case if the search cost is constant, because the cost imposed by time discounting is heterogeneous among agents. In a setup with some similarities to ours, Lien [2006] evaluates the role of a limited number of interviews in college admission outcomes. In his model, colleges choose which students to interview based on noisy public signals about students' match quality. Depending on how informative the public signals are, high-ability students may "fall through the cracks," due to the fact that lower-ranked colleges shy away from interviewing high-ranked students to avoid "wasting" interviews. In a related paper, Kadam [2014] considers a model in which firms engage in costly interviews to learn the value of a binary "fitness" of students. Firms, in equilibrium, spread their interviews among "star" candidates, medium-ranked students, and "safe bets."

Lee and Schwarz [2016] also consider firms' (or colleges') screening decisions. Similar to our model, the degree to which workers receive interviews from multiple firms has an impact on the efficiency of the match. As a result, a firm's expected payoff depends not only on the number of interviews its workers receive, but also the identities of the firms interviewing these workers. Differently from our model, however, workers' preferences are uniformly drawn and therefore there is no role for that in firms' screening strategies. In a model in which workers and firms share the surplus generated by their matches, Josephson and Shapiro [2016] show that costly screening by firms may prevent efficient matching because potentially good candidates are not interviewed, to avoid competition from more productive firms. Other papers, such as Che and Koh [2016] and Hafalir et al. [2016], also consider the role of competition between colleges when selecting students.

Ely and Siegel [2013] evaluate how the revelation of interviewing decisions by other firms affect equilibrium outcomes. They show that when firms can observe other firm's interviewing decisions, that information can be better exploited by the most-preferred firm. This happens because, when facing multiple offers, workers accept the one from the most preferred firm. This makes the choice of interviewing by less preferred firms very informative, leading the top firm to also interview. In our model, although firms can observe each other's aggregate screening decisions, that is not the case for individual students. The choice of how many students to screen, however, does have a similar effect in the less desirable school or college, which have to anticipate the fact that the more the other school screens, the more likely it will be that they will send offers to the same student.

Chade et al. [2014] consider the effects that application costs have on students and colleges' behavior in equilibrium. Given these costs, students face a portfolio choice problem in their application decision. In their model, both students and colleges act strategically and make decisions under uncertainty. In equilibrium, there may not be assortative matching, because weaker students may apply more aggressively, while smaller but weaker colleges may impose higher standards.

One important characteristic that distinguishes our model from the ones above is that colleges use the school that students come from as an endogenous signal of the ability of its applicants. Arrow [1973] introduced the concept of higher education as a "filter," where the screening performed by colleges when selecting applicants is used by firms as a signal of the student's ability. His baseline model, like ours, makes the simplifying assumption that education does not change the ability of a student, but instead that the screening process produces a signal about that student's unobservable ability. In a dynamic model in which the quality of past cohorts is partially used to produce rankings over schools, Herresthal [2017] shows that the informativeness of the rankings are enhanced, in a steady-state equilibrium, if a greater proportion of the selection of students is merit-based and when the costs of attending non-local schools are reduced. Conley and Önder [2014] evaluate empirically how much information about the productivity of economists is obtained by observing the ranking of the department where he or she obtained his or her PhD. Perhaps surprisingly, they find that the ranking of a student in a program is often a better predictor of future performance than the ranking of the department itself. Nevertheless, Baghestanian and Popov [2014] show that the publication market values the signals from the Alma mater of, and the position held by, the author.

Still, regarding the use of the signaling value of the school of origin, one aspect that is not explored in our paper is how schools may strategically inflate students' grades, with the objective of placing them in better colleges or jobs. Chan et al. [2007] present a model in which schools inflate grades to oversell their students in equilibrium. This grade inflation, however, reduces matching efficiency and in fact harms schools. Popov and Bernhardt [2013] present empirical evidence of grade inflation and similar theoretical results.

The remainder of the paper proceeds as follows. Section 2 introduces the model. The equilibrium choices for the schools and for the colleges are presented and analyzed in sections 3 and 4, respectively. We analyze the effects of students' preferences on outcomes and welfare in section 5, and analyze incentives and equilibrium behavior by the students in section 6. All proofs are relegated to the appendix.

2 Model

Our education system consists of two schools: S_1 and S_2, and two colleges: C_A and C_B. Schools S_1 and S_2 are entry-level schools, that is, there is no pre-requisite for attending them, whereas in order to be accepted at colleges C_A or C_B, students must have attended a school. Institutions have limited capacity: schools S_1 and S_2 can accept masses of at most $q > 0$ students, and colleges C_A and C_B can accept at most $Q > 0$ students each. Students can be acquired from three sources. One source is an *"external" pool* with a continuum of students, with total mass 1, where a fraction $\alpha > 0$ of them are deemed to be high-ability students. The other two are *"internal" pools* with an arbitrarily large continuum of students for the two levels of education. Those students from the internal pools have a known ability level, with very small variation between them.[1] Each individual student

[1]One way to interpret the internal pools is to think of them as students who come from earlier stages of the institution itself (for example, students from a college's high school), or those who have a diploma from the standard education system, while those from the external pool are international students, for which

has zero mass.

A share σ, where $0 \leqslant \sigma < \frac{1}{2}$, of the students prefer school S_1 over school S_2. That is the majority of students prefer school S_2 over S_1. The remaining prefer S_2 over S_1. All students prefer college C_A over C_B. There is no cost for applying to those institutions. Schools and colleges have linear preferences on the proportion of high-ability students among those that they accept, but strictly prefer filling seats with students from the internal pool than with those who did not pass the screening test. Similarly, we assume that colleges C_A and C_B prefer students from the internal pool over students from schools S_1 and S_2 who are not identified as being high ability. The value of these students from internal pools is normalized to zero. The result of these assumptions is that schools and colleges' objectives will consist of selecting as many high-ability students as possible, subject to cost and strategic considerations, and then filling the remaining seats with students from the internal pool.

We consider a two-stage admission process. In the first stage, students can make costless applications to schools S_1 and/or S_2. Given the external pool of applicants, schools choose how much screening to perform. That choice consists of setting a budget for screening that targets a specific expected number of high-ability students, given the knowledge of the overall proportion of high-ability students α. This is done by, for example, giving exams. Each high-ability student from the pool of applicants is equally likely to be identified, and this identification is independent between schools. The cost that a school incurs in identifying a mass λ of high-ability students from a pool of η applicants with a mass P_h of high-ability students is the following, where $\kappa > 0$:

$$C(\lambda) = \kappa \int_0^{\lambda} \frac{\eta x}{P_h - x} dx.$$

The screening process can be thought of as a search for students, examining them one by one to evaluate whether the student is of high-ability. That search is increasingly difficult the smaller the proportion of students with high ability is, which is represented by the term $\frac{P_h - x}{\eta}$ in the denominator. We assume that the marginal cost of screening is also increasing in the absolute number of high-ability students that are identified, leading therefore to the expression above. Schools are free to combine students who are selected through screening and the internal pool. We assume that the criterion set for establishing

there is a larger uncertainty about the quality of their education. The value of α can be considered as a function of the overall quality of the students in the internal pool: it can be the proportion of students in the external pool who have an ability that is unambiguously higher than those in the internal pool.

a student as high ability to be selective enough such that those are always preferred over those from the internal pool. The policy for a school S_i consists of choosing an expected number of high-ability students from the external pool to select by screening, λ_i, sending them offers and filling the remaining seats (if any) with students from the internal pool. After both schools perform that screening process simultaneously, they will proceed to send offers to students (also simultaneously). Students who do not receive any offer will remain unmatched. Students who receive offers from one or both schools may accept at most one of them.

In the second stage, after the matching process from the schools ends, students apply to colleges, which perform the same kind of screening that schools performed over the external pool. In the case of colleges, however, the screening is made over the set of students who attended schools S_1 and S_2. Since colleges can perfectly and costlessly observe which school students come from, they can make independent screening decisions over the students who come from each one of the two schools. We assume that although schools may have an impact on a student's abilities, this impact is uniform across schools and cohorts. As a result, students who are deemed to be high ability in the first stage are still the ones who are high ability in the second stage. After simultaneously screening students, colleges send offers to the high-ability students identified. Again, students who do not receive any offer will remain unmatched. Students who receive offers from one or both colleges may accept at most one of them. If there are unfilled seats, those are filled with students from the internal pools.

3 School Admission

We consider the situation where all students apply to both institutions, in each level. Students who receive more than one offer can choose which one to accept, if any. Given the values of λ_1 and λ_2 and the fact that every high-ability student has the same probability of being identified as such, independently, by each school, the mass of students who are identified by both schools as being high ability is $\frac{\lambda_1 \lambda_2}{\alpha}$. For simplicity, we assume that the size of the internal pools of students is large enough so that the mass of students who receive offers from both schools is insignificant. If students follow their preferences,[2] a proportion σ of those students who receive offers from both schools will accept school S_1, and the remaining will accept school S_2.

[2] In section 6 we test that assumption when students behave strategically.

Let H_1 and H_2 be the expected number of high-ability students admitted by schools S_1 and S_2. Then:

$$H_1 = \lambda_1 - (1-\sigma)\,\frac{\lambda_1\lambda_2}{\alpha}, \text{ and } H_2 = \lambda_2 - \sigma\frac{\lambda_1\lambda_2}{\alpha}.$$

Since the number of students with high ability in the external pool is α and the total mass of that pool is one, $\eta = 1$ and $P_h = \alpha$ and utilities for the schools are then:

$$\begin{aligned} U_1 =& \lambda_1 - (1-\sigma)\,\frac{\lambda_1\lambda_2}{\alpha} - \kappa\int_0^{\lambda_1}\frac{x}{\alpha - x}dx, \text{ and} \\ U_2 =& \lambda_2 - \sigma\frac{\lambda_1\lambda_2}{\alpha} - \kappa\int_0^{\lambda_2}\frac{x}{\alpha - x}dx. \end{aligned}$$

Schools' best-response functions are as follows:

$$\lambda_1 = \alpha\left(\alpha - \lambda_2\left(1-\sigma\right)\right), \text{ and } \lambda_2 = \frac{\alpha\left(\alpha - \lambda_1\sigma\right)}{\alpha(1+\kappa) - \lambda_1\sigma}.$$

Notice that $\frac{\partial\lambda_1}{\partial\lambda_2} < 0$ and $\frac{\partial\lambda_2}{\partial\lambda_1} < 0$, that is, schools' screenings are strategic substitutes. Therefore, the maximum value for λ_1 is produced when $\lambda_2 = 0$, in which case $\lambda_1 = \frac{\alpha}{1+\kappa} < \alpha$. Screening decisions are thus always interior. The same holds for λ_2 . The unique Nash Equilibrium strategy profile is therefore:

$$\begin{aligned} \lambda_1^* =& \frac{\kappa(\kappa+2) + \sigma(\sigma+1) - A}{2\sigma(\kappa+\sigma)}\alpha, \text{ and} \\ \lambda_2^* =& \frac{\kappa(\kappa+2) + (\sigma-3)\sigma + 2 - A}{2(1-\sigma)(\kappa+1-\sigma)}\alpha \end{aligned}$$

where:

$$A = \sqrt{\kappa^4 + 4\kappa^3 + 2\kappa^2(2-\sigma(1-\sigma)) + 4\kappa\sigma(1-\sigma) + (1-\sigma)^2\sigma^2}.$$

Then the equilibrium numbers of high-ability students in each school are:

$$\begin{aligned} H_1^* =& \frac{\alpha\left(\kappa^2 - A - \sigma(1-\sigma)\right)\left(\kappa(\kappa+2) - A + \sigma(\sigma+1)\right)}{4\sigma(\sigma-1-\kappa)(\kappa+\sigma)}, \text{ and} \\ H_2^* =& \frac{\alpha\left(\kappa^2 - A - \sigma(1-\sigma)\right)\left(\kappa(\kappa+2) - A + (\sigma-3)\sigma + 2\right)}{4(1-\sigma)(\sigma-1-\kappa)(\kappa+\sigma)}. \end{aligned}$$

We will assume that the value of q (the capacity of schools S_1 and S_2) is high enough such that the solutions are interior.

Assumption 1. *The capacity of schools are high enough such that $q \geqslant \alpha$.*

The purpose of the assumption is simply to guarantee that no school will, in equilibrium, have only high-ability students.[3] As discussed in section 4, when a school only has high-ability students, the analysis of colleges' decisions becomes trivial, not allowing for most of the insights that we obtain in what follows.

Proposition 1. *For the schools' admission process:*

(i) the less popular school S_1 admits less high-ability students than the more popular school S_2 ($H_1^* < H_2^*$), and screens less ($\lambda_1^* < \lambda_2^*$); and when screening become costly,

(ii) both schools screen less ($\frac{\partial \lambda_1^*}{\partial \kappa} < 0$ and $\frac{\partial \lambda_2^*}{\partial \kappa} < 0$),

(iii) the more popular school S_2 admits less high-ability students ($\frac{\partial H_2^*}{\partial \kappa} < 0$), and

(iv) the less popular school S_1 admits more high-ability students if and only if the screening cost is low: there exists a $\kappa^* > 0$ such that:

$$\begin{cases} \kappa < \kappa^* & \textit{implies } \frac{\partial H_1^*}{\partial \kappa} > 0, \textit{ and} \\ \kappa > \kappa^* & \textit{implies } \frac{\partial H_1^*}{\partial \kappa} < 0. \end{cases}$$

Part (i) of proposition 1 shows that the model is well-behaved and produces natural results. Since more students prefer school S_2 over S_1, the expected marginal benefit from screening is higher for the former, since a greater proportion of the students who receive two offers will go there. As a result, S_2 will screen more and obtain more high-ability students in equilibrium. Also, part (ii) is also natural: higher costs of screening should lead to less screening by both schools.

Perhaps the most intriguing result is the one in part (iv). It shows that when the cost of screening is low enough, an increase in the cost of screening would lead to an increase in the number of high-ability students acquired by S_1, even though there is also a reduction on the amount of screening done by that school. The reason for this is that the reduction in screening by S_1, and its consequent reduction in expected number of high ability students acquired, is more than compensated by the reduction in the number of students who also receive an offer from S_2, since a majority of them will reject the offer from S_1. This only happens, however, when the total amount of screening performed by both schools is enough to make the number of students with two offers high. This only happens when the cost of screening is low.

[3]Technically, for our purposes, it would be sufficient to assume that $q > H_2^*$.

A consequence of this result is that as the use of technology allows for a reduction in the cost of screenings, there may be a point at which it will lead to an increase in the competition for each high-ability student, in a way that schools which are lower-ranked will, despite being able to screen more students, see a smaller number of them accepting their offers.

4 College Admission

As with the schools, colleges first screen the students simultaneously and then send offers, also simultaneously. Since colleges can perfectly observe which school among S_1 and S a student comes from, they can treat them as two different pools and can make separate choices as to how much to screen from each. Let λ_i^1 and λ_i^2 denote college C_i's choice of how many high-ability students to search for among students from schools S_1 and S_2, respectively. Since the number of students in school S_1 is q and the number of high-ability students there is H_1^*, the values of η and P_h for the cost function in that school are q and H_1^*, respectively. Making the same consideration for school S_2, the search cost for each college is, therefore, as follows, where $\delta > 0$:

$$\begin{aligned} Cost_{C_A} =& \delta \left(\int_0^{\lambda_A^2} \frac{qx}{H_2^* - x} dx + \int_0^{\lambda_A^1} \frac{qx}{H_1^* - x} dx \right), \text{ and} \\ Cost_{C_B} =& \delta \left(\int_0^{\lambda_B^2} \frac{qx}{H_2^* - x} dx + \int_0^{\lambda_B^1} \frac{qx}{H_1^* - x} dx \right). \end{aligned}$$

Notice that since $H_2^* > H_1^*$, the marginal cost for finding high-ability students from those who attended school S_2 grows, initially, at a lower rate.

If we relaxed Assumption 1 and allowed for school S_2 to be binding then the solution for the colleges' decisions would be simple: college C_A will accept as many high-ability students as possible from S_2. If there is still capacity left at college C_A then we are back to a problem with a single pool. Otherwise, college C_B will accept the remaining students from S_2 and will then screen from the remaining single pool in S_1.

Notice that when we say that a solution is "binding" we mean that the number of high-ability students admitted equals the capacity of the college.

4.1 Interior solutions

First, we consider the case in which the equilibrium number of high-ability students acquired by both colleges is smaller than Q. Since every student prefers college C_A over C_B, students who receive offers from both colleges will choose to go to C_A. The number of high-ability students that the colleges acquire are therefore:

$$H_A = \lambda_A^1 + \lambda_A^2 \quad \text{and} \quad H_B = \lambda_B^1 - \tfrac{\lambda_B^1 \lambda_A^1}{H_1^*} + \lambda_B^2 - \tfrac{\lambda_B^2 \lambda_A^2}{H_2^*} \ .$$

Colleges' payoffs are:

$$U_A = \lambda_A^1 + \lambda_A^2 - \delta \left(\int_0^{\lambda_A^2} \frac{qx}{H_2^* - x} dx + \int_0^{\lambda_A^1} \frac{qx}{H_1^* - x} dx \right), \text{ and}$$

$$U_B = \lambda_B^1 - \frac{\lambda_B^1 \lambda_A^1}{H_1^*} + \lambda_B^2 - \frac{\lambda_B^2 \lambda_A^2}{H_2^*} - \delta \left(\int_0^{\lambda_B^2} \frac{qx}{H_2^* - x} dx + \int_0^{\lambda_B^1} \frac{qx}{H_1^* - x} dx \right).$$

Colleges' best-response functions are as follows:

$$\left(\lambda_A^1, \lambda_A^2, \lambda_B^1, \lambda_B^2 \right) = \left(\frac{H_1^*}{1 + \delta q}, \frac{H_2^*}{1 + \delta q}, \frac{H_1^* \left(H_1^* - \lambda_A^1 \right)}{H_1^* \left(1 + \delta q \right) - \lambda_A^1}, \frac{H_2^* \left(H_2^* - \lambda_A^2 \right)}{H_2^* \left(1 + \delta q \right) - \lambda_A^2} \right).$$

The unique Nash Equilibrium strategy profile is therefore:

$$\left(\lambda_A^{1*}, \lambda_A^{2*}, \lambda_B^{1*}, \lambda_B^{2*} \right) = \left(\frac{H_1^*}{1 + \delta q}, \frac{H_2^*}{1 + \delta q}, \frac{H_1^*}{2 + \delta q}, \frac{H_2^*}{2 + \delta q} \right).$$

Thus, the equilibrium numbers of high-ability students are:

$$\left(H_A^*, H_B^* \right) = \left(\frac{H_1^* + H_2^*}{1 + \delta q}, \frac{\delta q \left(H_1^* + H_2^* \right)}{\left(1 + \delta q \right) \left(2 + \delta q \right)} \right).$$

Notice that $\frac{H_A^*}{H_B^*} = 1 + \frac{2}{\delta q}$. That is, when the capacity of the colleges is large enough such that none of the solutions are binding, the relative advantage that college C_A enjoys over C_B is independent of key parameters of the schools' matching stage, except for the capacities of the schools (q).

Especially interesting is the fact that the ratio is independent of students' preferences between schools. For example, when σ is close to zero, implying that a large majority of students prefer school S_2 over S_1, college C_A is able to screen those students from S_2, and successfully acquire them, at a low cost. This, however, does not translate into an advantage in terms of equilibrium cohort composition over C_B when compared to a situation in which high-ability students are more evenly distributed between the two schools.

4.2 Only college C_A binding

For college C_A, since every student will accept its offers, the optimal decision is independent of college C_B's decision. Moreover, the marginal gain from screening high-ability students is always 1, since every screened student will accept that college's offer. Since the marginal benefit is constant, college C_A will screen from S_1 and S_2 while keeping their marginal costs equal:

$$\frac{q\lambda_A^{1*}}{H_1^* - \lambda_A^{1*}} = \frac{q\lambda_A^{2*}}{H_2^* - \lambda_A^{2*}}.$$

We are interested in evaluating the case in which this will be binding, that is, $\lambda_A^{1*} + \lambda_A^{2*} = Q$. We can, therefore, replace λ_A^{2*} by $Q - \lambda_A^{1*}$:

$$\frac{q\lambda_A^{1*}}{H_1^* - \lambda_A^{1*}} = \frac{q\left(Q - \lambda_A^{1*}\right)}{H_2^* - \lambda_A^{2*}}.$$

Since all high-ability students screened by C_A accept its offers, the expressions for the number of those acquired from S_1 and S_2 are:

$$H_A^{1*} = \lambda_A^{1*} = \frac{H_1^* Q}{H_1^* + H_2^*}, \text{ and } H_A^{2*} = \lambda_A^{2*} = \frac{H_2^* Q}{H_1^* + H_2^*}.$$

That is, college C_A will always screen from each school proportional to that school's share of the overall high-ability students in schools. That leads to the following:

Proposition 2. *Let H_A^{1*} and H_A^{2*} be the masses of students from S_1 and S_2, respectively, accepted at C_A, and assume that $Q < H_A^*$ (that is, school C_A's capacity is binding). Then:*

(i) when both colleges expand, the more popular college C_A admits additionally more students from the more popular school S_2 than the less popular school S_1($\frac{\partial H_A^{2*}}{\partial Q} > \frac{\partial H_A^{1*}}{\partial Q} > 0$), but

(ii) when screening becomes costly, the more popular college C_A admits more students from the more popular school S_2, but less students from the less popular school S_1 ($\frac{\partial H_A^{1*}}{\partial \kappa} > 0$ and $\frac{\partial H_A^{2*}}{\partial \kappa} < 0$).

That is, if the capacities of colleges are reduced, then there is a larger reduction of students from S_1 matched to C_A than that of those from S_2, even though their capacities remain equal to each other. One way to interpret this result is that when both colleges become more selective, being a student at a more competitive school becomes more important for their chances of getting into the better college, even while controlling for the overall increase in competition for colleges. This happens because the proportion of students from the top school screened by C_A is higher, due to the convexity of the screening cost.

Similarly, as the screening technology used by schools becomes cheaper, the absolute number of students from S_1 admitted to college C_A is reduced. This happens because the impact that a reduction in κ has is larger, at the margin, on S_2, resulting in a relative gain for that school when compared to S_1. This in turn increases the attractiveness of screening students from S_2, since a higher proportion of high-ability students there reduce the screening cost from them.

Consider now college C_B. Its optimal screenings will be such that the marginal utilities from both schools is zero:

$$\begin{aligned}\frac{\partial U_B}{\partial \lambda_B^1} =& 1 - \frac{\lambda_A^1}{H_1^*} - \frac{\delta q \lambda_B^1}{H_1^* - \lambda_B^1} = 0, \text{ and}\\ \frac{\partial U_B}{\partial \lambda_B^2} =& 1 - \frac{\lambda_A^2}{H_2^*} - \frac{\delta q \lambda_B^2}{H_2^* - \lambda_B^2} = 0.\end{aligned}$$

In the case where $Q < H_A^*$, we set $\lambda_A^1 = \frac{H_1^* Q}{H_1^* + H_2^*}$ and $\lambda_A^2 = \frac{H_2^* Q}{H_1^* + H_2^*}$:

$$1 - \frac{\frac{H_1^* Q}{H_1^* + H_2^*}}{H_1^*} - \frac{\delta q \lambda_B^{1*}}{H_1^* - \lambda_B^{1*}} = 0, \text{ and}$$

$$1 - \frac{\frac{H_2^* Q}{H_1^* + H_2^*}}{H_2^*} - \frac{\delta q \lambda_B^{2*}}{H_2^* - \lambda_B^{2*}} = 0.$$

which yields:

$$\begin{aligned}1 - \frac{Q}{H_1^* + H_2^*} - \frac{\delta q \lambda_B^{1*}}{H_1^* - \lambda_B^{1*}} &= 0, \text{ and}\\ 1 - \frac{Q}{H_1^* + H_2^*} - \frac{\delta q \lambda_B^{2*}}{H_2^* - \lambda_B^{2*}} &= 0.\end{aligned}$$

Solving for λ_B^{1*} and λ_B^{2*}, we obtain:

$$\left(\lambda_B^{1*}, \lambda_B^{2*}\right) = \left(\frac{\frac{Q}{H_1^*+H_2^*} - H_1^*}{\frac{Q}{H_1^*+H_2^*} - \delta q - 1}, \frac{\frac{Q}{H_1^*+H_2^*} - H_2^*}{\frac{Q}{H_1^*+H_2^*} - \delta q - 1} \right).$$

Therefore:

$$\begin{aligned}
H_B^{1*} =& \lambda_B^{1*} - \frac{\lambda_B^{1*}\lambda_A^{1*}}{H_1^*} = \left(\frac{\frac{Q}{H_1^*+H_2^*} - H_1^*}{\frac{Q}{H_1^*+H_2^*} - \delta q - 1} \right)\left(1 - \frac{Q}{H_1^* + H_2^*}\right), \\
H_B^{2*} =& \lambda_B^{2*} - \frac{\lambda_B^{2*}\lambda_A^{2*}}{H_2^*} = \left(\frac{\frac{Q}{H_1^*+H_2^*} - H_2^*}{\frac{Q}{H_1^*+H_2^*} - \delta q - 1} \right)\left(1 - \frac{Q}{H_1^* + H_2^*}\right), \text{ and} \\
H_B^* =& H_B^{1*} + H_B^{2*} = \left(\frac{\frac{2Q}{H_1^*+H_2^*} - H_1^* - H_2^*}{\frac{Q}{H_1^*+H_2^*} - \delta q - 1} \right)\left(1 - \frac{Q}{H_1^* + H_2^*}\right).
\end{aligned}$$

Notice that both H_A^* and H_B^* depend on the result of the schools' matching stage through only two values: q and $H_1^* + H_2^*$. That is, the number of high-ability students obtained by each college depends only on the total number available, and not on how they are distributed between the schools.

Proposition 3. *Let $Q < H_A^*$. When colleges expand their capacities, the less popular college C_B screens less students from both schools, that is,*

$$\frac{\partial \lambda_B^{1*}}{\partial Q} < 0 \text{ and } \frac{\partial \lambda_B^{2*}}{\partial Q} < 0.$$

Notice that $\frac{\partial \lambda_A^{1*}}{\partial Q} > 0$ and $\frac{\partial \lambda_A^{2*}}{\partial Q} > 0$. That is, when the colleges' capacities increase, the impact on their screening is opposite. This is due to the fact that when Q increases and college C_A is still binding, the number of high-ability students that are screened and acquired by C_A increases by the same amount. This leads to a reduction on the marginal gain from screening for college C_B, since the increases in λ_A^1 and λ_A^2 imply a higher number of students who will also receive an offer from C_A (and therefore reject C_B). The result is that college C_B's optimal amount of screening from both schools will be reduced.

Another property of school C_B's outcome is that the proportion of high-ability students changes differently from school C_A when Q changes:

Proposition 4. *Let $Q < H_A^*$. When both colleges expand, the share of high-ability students in the more popular college C_A remains unchanged but that for the less popular college C_B decreases, or*

$$\frac{\partial}{\partial Q}\frac{H_B^*}{Q} < 0 \textit{ and } \frac{\partial}{\partial Q}\frac{H_A^*}{Q} = 0.$$

That is, when capacities of schools decrease, school C_B will increase the average quality of their students. The reason is that as Q becomes smaller, the number of students receiving offers from C_A decreases, decreasing the competition that C_B faces from C_A.

4.3 Both colleges binding

Here, we consider the case in which both colleges fill their capacities, in equilibrium, with high-ability students. Although there is no longer any question about the proportion of high-ability students among the colleges, the question of how the distribution of students from both schools into the colleges responds to the parameters is still of interest.

The condition for college C_B to be binding is:

$$H_B^* \geqslant Q.$$

That is:

$$\left(\frac{\frac{2Q}{H_1^*+H_2^*} - H_1^* - H_2^*}{\frac{Q}{H_1^*+H_2^*} - \delta q - 1}\right)\left(1 - \frac{Q}{H_1^* + H_2^*}\right) \geqslant Q.$$

Solving for q yields:

$$\delta q < \frac{H_1^* + H_2^*}{Q} - \frac{2(1-Q)}{\left(H_1^* + H_2^*\right)^2} - \frac{2-Q}{H_1^* + H_2^*}.$$

Since $Q < H_1^* + H_2^*$, the right-hand side of the expression above being positive is equivalent to:

$$Q < \frac{\left(H_1^* + H_2^*\right)^2}{2 + \left(H_1^* + H_2^*\right)}.$$

That is, for college C_B to be binding, the excess supply of high-ability students from the schools must be high enough compared to the capacities of the colleges. The optimality condition is for marginal utilities from screening from both schools to be equalized:

$$\frac{\partial U_B}{\partial \lambda_B^1} = \frac{\partial U_B}{\partial \lambda_B^2}, \text{ and}$$

$$1 - \frac{\lambda_A^{1*}}{H_1^*} - \frac{\delta q \lambda_B^{1*}}{H_1^* - \lambda_B^{1*}} = 1 - \frac{\lambda_A^{2*}}{H_2^*} - \frac{\delta q \lambda_B^{2*}}{H_2^* - \lambda_B^{2*}}.$$

Since college C_B is binding, λ_B^{1*} and λ_B^{2*} are such that, after taking into consideration the simultaneous offers from C_A, the number of students who accept offers from C_B is Q. When we also replace λ_A^{1*} and λ_A^{2*} by their binding values, we get:

$$\begin{aligned} 1 - \frac{\frac{H_1^* Q}{H_1^*+H_2^*}}{H_1^*} - \frac{\delta q \lambda_B^{1*}}{H_1^* - \lambda_B^{1*}} &= 1 - \frac{\frac{H_2^* Q}{H_1^*+H_2^*}}{H_2^*} - \frac{\delta 0 q \lambda_B^{2*}}{H_2^* - \lambda_B^{2*}}, \text{ and} \\ \lambda_B^{1*} - \frac{\lambda_B^{1*} \frac{H_1^* Q}{H_1^*+H_2^*}}{H_1^*} + \lambda_B^{2*} - \frac{\lambda_B^{2*} \frac{H_2^* Q}{H_1^*+H_2^*}}{H_2^*} &= Q. \end{aligned}$$

Hence, we have

$$\left(\lambda_B^{1*}, \lambda_B^{2*}\right) = \left(\frac{H_1^* Q}{H_1^* + H_2^* - Q}, \frac{H_2^* Q}{H_1^* + H_2^* - Q}\right).$$

The number of students from each school is, therefore, the same as in college C_A:

$$\left(H_B^{1*}, H_B^{2*}\right) = \left(\frac{H_1^* Q}{H_1^* + H_2^*}, \frac{H_2^* Q}{H_1^* + H_2^*}\right).$$

Thus, when both colleges are binding, both will acquire the same proportion of high-ability students from each school. The difference is that this will take place at a higher cost for college C_B. The results below show that the responses in college C_B to changes in schools' screening costs and colleges' capacities are the same as that for college C_A.

Proposition 5. *Let* $Q < H_A^*$ *and* $Q < \frac{\left(H_1^*+H_2^2\right)^2}{2+H_1^*+H_2^*}$. *Then:*

(i) when both colleges expand their capacities, both colleges admit additionally more students from the more popular school S_2 *than the less popular school (*$\frac{\partial \lambda_A^{2*}}{\partial Q} > \frac{\partial \lambda_A^{1*}}{\partial Q} > 0$ *and* $\frac{\partial \lambda_B^{2*}}{\partial Q} > \frac{\partial \lambda_B^{1*}}{\partial Q} > 0$*), and*

(ii) when screening becomes more costly, both colleges admit more students from the more popular school S_2*, but less students from the less popular school* S_1 *(*$\frac{\partial H_A^{1*}}{\partial \kappa} > 0$, $\frac{\partial H_B^{1*}}{\partial \kappa} > 0$, $\frac{\partial H_A^{2*}}{\partial \kappa} < 0$, *and* $\frac{\partial H_B^{2*}}{\partial \kappa} < 0$*).*

5 Welfare analysis and the role of preferences

In this section, we evaluate how students' preferences between schools affect schools' outcomes and student welfare. Variations in students' preferences are represented by changes in the value of σ. For the purpose of welfare analysis, instead of making cardinal assumptions, we focus on the number of students who are admitted to their most preferred school and/or college. Moreover, we will focus on the high-ability students. [4]

5.1 Schools' matching

The main results regarding how students' preferences affect outcomes are driven by their effect on schools' equilibrium screening and intake of high-ability students. Remember that we assume, without loss of generality, that $0 < \sigma < \frac{1}{2}$. Therefore, an increase in σ represents a reduction in the aggregate preference that students have for school S_2.

Proposition 6. *As preferences between schools become more heterogeneous,*

(i) more students receive offers from both schools ($\frac{\partial \lambda_1^* \lambda_2^*}{\partial \sigma} > 0$*),*

(ii) the less popular school admits more high-ability students ($\frac{\partial H_1^*}{\partial \sigma} > 0$*) and the more popular school admits less (*$\frac{\partial H_2^*}{\partial \sigma} < 0$*), and*

(iii) both schools admit more high-ability students ($\frac{\partial \left(H_1^* + H_2^*\right)}{\partial \sigma} > 0$*).*

Item (ii) of proposition 6 is intuitive: as more students prefer school S_1, the equilibrium number of high-ability students in S_1 increases and that in S_2 decreases.

Item (i) is less obvious. Changes in the value of $\lambda_1^* \lambda_2^*$ reflect changes in the number of students who receive offers from both schools. Students' preferences affect schools' screening choices by changing the expected benefit from screening. As σ increases, the marginal gain from screening increases for school S_1 and decreases for S_2. More specifically, below are the changes in the marginal gain from screening driven by changes in preferences:

$$\frac{\partial}{\partial \sigma} \frac{\partial U_1}{\partial \lambda_1} = \frac{\lambda_2}{\alpha} \text{ and } \frac{\partial}{\partial \sigma} \frac{\partial U_2}{\partial \lambda_2} = -\frac{\lambda_1}{\alpha}.$$

By proposition 1, $\lambda_2^* > \lambda_1^*$. Therefore, starting from equilibrium values, the increase in the marginal gain from the screening of school S_1 has a larger magnitude than the

[4] This focus is justified by the fact that the number of students from internal pools who receive offers from two schools or colleges is insignificant, and as a result the analysis of the number of them who receive an offer from their most preferred institution is not of special interest.

decrease in S_2. Therefore, the overall gain from screening increases, leading to a higher overall amount of screening by both schools. As a result, the total number of admitted high-ability students also increases, as shown in item (iii).

The number of students who have school S_1 as their most preferred school and receive an offer from S_1 is $\sigma\lambda_1^*$. The number of those who have school S_2 as their most preferred school and receive an offer from S_2 is $(1-\sigma)\,\lambda_2^*$. All the other high-ability students who are admitted are matched to their second choice. So the number of students who are matched to their most preferred schools is:

$$\mathcal{H}^{1,2} = \sigma\lambda_1^* + (1-\sigma)\,\lambda_2^*.$$

Proposition 7. *As preferences become more heterogeneous, there will be a lower number and proportion of high-ability students matched to their most preferred school. Let $\mathcal{H}^{1,2}$ be the number of students who are admitted to their most preferred school among S_1 and S_2. Then,*

$$\frac{\partial \mathcal{H}^{1,2}}{\partial \sigma} < 0 \text{ and } \frac{\partial}{\partial \sigma}\frac{\mathcal{H}^{1,2}}{H_1^* + H_2^*} < 0.$$

That is, as preferences become less correlated, the number (and proportion) of high-ability students who are matched to their top choice decreases. At first sight this may seem unintuitive. After all, when preferences are less correlated there is less competition between students for the schools' seats. Moreover, as shown in Proposition 6, there is an overall increase in the total amount of screening, so that more students receive offers from both schools. To better understand the source of that result, it is useful to disentangle the two effects that the shift in students' preferences has on $\mathcal{H}^{1,2}$:

$$\frac{\partial \mathcal{H}^{1,2}}{\partial \sigma} = \underbrace{\sigma\frac{\partial \lambda_1^*}{\partial \sigma} + (1-\sigma)\,\frac{\partial \lambda_2^*}{\partial \sigma}}_{\text{Screening change effect}} + \underbrace{\lambda_1^* - \lambda_2^*}_{\text{Screening gap effect}}.$$

The *screening change effect* relates to the fact that a change in σ leads to an increase in λ_1^* and a decrease in λ_2^*. The former produces an increase in the number of students who prefer S_1 and receive an offer from S_1, while the latter produces a reduction in the number of students who prefer S_2 and receive an offer from S_2. The *screening gap effect*, on the other hand, relates to the fact that the increase in σ implies that more students prefer a school that screens fewer students than the one that screens more. The result shows that even when the screening change effect increases the number of students matched to their preferred school, this is dominated by the screening gap effect. In fact, since the screening cost is convex, it is natural that school S_1's increase in screening is not sufficient to accommodate every student who becomes interested in it.

5.2 Colleges' matching

The results from Proposition 6 produce some important effects on colleges' outcomes as well, outlined in the proposition below:

Proposition 8. *As preferences between schools become more heterogeneous, colleges admit more high-ability students, as long as their capacities are not binding. In other words, if C_A is not binding, $\frac{\partial H_A^*}{\partial \sigma} > 0$ and if C_B is not binding, $\frac{\partial H_B^*}{\partial \sigma} > 0$.*

Proposition 8 shows that, as opposed to schools, both colleges obtain a better set of students when preferences are more heterogeneous. This happens because colleges are able to "free ride" on schools' increased screening in reaction to the change in preferences.

Since students have common preferences between the colleges, the number of students who are matched to their top choice among them is simply the number of students matched to college C_A. We now proceed to evaluate overall welfare combining the two levels of education.

The proportion of high-ability students in schools S_1 and S_2 who are at their top choice among schools are $\frac{\sigma\lambda_1^*}{H_1^*}$ and $\frac{(1-\sigma)\lambda_2^*}{H_2^*}$, respectively. Since the number of students that college C_A admits from each of the schools equals the number of those who are screened, the total number of students who are matched to their top school and are then matched to C_A (their top choice among colleges) is:

$$\mathcal{H}^{A,B} = \lambda_A^{1*}\sigma\frac{\lambda_1^*}{H_1^*} + \lambda_A^{2*}\left(1-\sigma\right)\frac{\lambda_2^*}{H_2^*}.$$

When both C_A and C_B are non-binding, $\lambda_A^{1*} = \frac{H_1^*}{1+\delta q}$ and $\lambda_A^{2*} = \frac{H_2^*}{1+\delta q}$. Let $\mathcal{H}_1^{A,B}$ be the value of $\mathcal{H}^{A,B}$ in that case:

$$\mathcal{H}_1^{A,B} = \frac{\sigma\lambda_1^* + \left(1-\sigma\right)\lambda_2^*}{1+\delta q} = \frac{\mathcal{H}^{1,2}}{1+\delta q}.$$

When C_A is binding, $\lambda_A^{1*} = Q\frac{H_1^*}{H_1^*+H_2^*}$ and $\lambda_A^{2*} = Q\frac{H_2^*}{H_1^*+H_2^*}$. Let $\mathcal{H}_2^{A,B}$ be the value of $\mathcal{H}^{A,B}$ in that case:

$$\mathcal{H}_2^{A,B} = \frac{\sigma\lambda_1^* + \left(1-\sigma\right)\lambda_2^*}{H_1^*+H_2^*}Q = \frac{\mathcal{H}^{1,2}}{H_1^*+H_2^*}Q.$$

The effect that changes in preferences have on $\mathcal{H}^{A,B}$ is the same in both cases:

Proposition 9. *As preferences between schools becomes more heterogeneous, there will be a lower number and proportion of high-ability students matched to their most preferred school and college. Let $\mathcal{H}^{A,B}$ be the number of students who are admitted to their most preferred school and college. If $Q > H_B^*$ then:*

$$\frac{\partial \mathcal{H}^{A,B}}{\partial \sigma} < 0 \text{ and } \frac{\partial}{\partial \sigma} \frac{\mathcal{H}^{A,B}}{H_1^{A*} + H_2^{A*} + H_1^{B*} + H_2^{B*}} < 0.$$

The result above shows that although that change in preferences increases the absolute number of students matched to both colleges, and as a consequence of those matched to their top college C_A (Proposition 8), that increase is more than compensated by a decrease in the number of students who are matched to their most preferred school.

There is no need to analyze the case where C_B is binding, since that only involves changes in the number of students who get their second choices.

6 Strategic behavior

Up to this point, we assumed that students' behavior were simple: they always apply to both schools and colleges, and will always accept the offer from the most preferred institution. In this section, we evaluate whether students may deviate from this "truthful" behavior in order to obtain better matches.

There are two choices that students make and that may be framed in terms of truthful or untruthful behavior. The first is whether students apply to both schools and colleges. Since the value of attending any school or college is higher than not attending any, and there is no cost for application, "truthful" behavior implies students applying to all schools and/or colleges.

The second choice is made when students receive offers and may accept them or not. Here, truthful behavior would consist of always accepting some offer and, when facing multiple offers, accepting that of the most preferred option. In principle, students could benefit from not acting truthfully. For example, when facing two offers, a student could accept the offer from a less desirable school, with the objective of increasing the likelihood of being matched to a more preferred college later on.

The sequence of events is the following:

- $t = 1$: Students simultaneously apply to schools S_1 and/or S_2,

- $t = 2$: Schools simultaneously choose the values of λ_1 and λ_2 and screen for high-ability students from their pool of applicants,

- $t = 3$: Schools simultaneously send offers to students,

- $t = 4$: Students who received offers may accept at most one of them,

- $t = 5a, 5b, \ldots$: Schools that have not filled their capacities send additional offers, students who received offers may accept at most one of them, etc, as many times as they choose to.

- $t = 6$: Eligible students apply to colleges C_A and/or C_B,

- $t = 7$: Colleges costlessly observe the school which a student comes from and simultaneously screen for high-ability students from their pools of applicants,

- $t = 8$: Colleges simultaneously send offers to students,

- $t = 9$: Students who received offers may accept at most one of them,

- $t = 10a, 10b, \ldots$: Colleges which did not fill their capacities send additional offers, students who received offers may accept at most one of them, etc. as many times as they choose to.

Since we will consider preferences over expected utilities that combine schools and colleges, in this section we make cardinal assumptions over students' preferences. More specifically, we will assume that preferences are additively separable, as follows:

$$U\left(S_i, C_j\right) = v_i + v_j, \text{ where } i \in \{1, 2\} \text{ and } j \in \{A, B\}.$$

Utilities obtained by being matched to each college and to each school are common among students who have the same ordinal preference between them. We denote by v_i^{ϕ} the cardinal utility that a student with the ordinal preference ϕ derives from being matched to the school or college C_i. That is, students who prefer school S_1 over S_2 derive utilities $v_1^{1 \succ 2}$ and $v_2^{1 \succ 2}$ from being matched to those schools, respectively, where $v_1^{1 \succ 2} > v_2^{1 \succ 2}$. Moreover, those who prefer school S_2 over S_1 derive utilities $v_1^{2 \succ 1}$ and $v_2^{2 \succ 1}$, where $v_1^{2 \succ 1} < v_2^{2 \succ 1}$. Utilities obtained from colleges C_A and C_B are common. The solution concept that we use is the Perfect Bayesian Equilibrium (PBE). Although in principle the number of strategies that students could have are large and complex, many of them can be easily eliminated from consideration in equilibrium.

First, notice that if a high-ability student receives an offer in period $t = 3$ or $t = 8$, she will not receive an offer from another institution later in steps $t = 5a, 5b, \ldots$ or $t = 10a, 10b, \ldots$, respectively. This is the case because schools and colleges send offers to all high-ability students that they identify through screening, and fill their remaining seats with students from an internal pool. The result of that is pointed out in the remark below:

Remark 1. In equilibrium, high-ability students are matched to schools and colleges only by the end of periods $t = 4$ and $t = 9$, respectively. Therefore, every offer that is accepted by high-ability students is accepted in those periods.

The next remark relates to whether students will apply to both schools and colleges in steps $t = 1$ and $t = 6$. Since each individual student has mass zero, the impact that they have on the value of the probability of being accepted into a school or college that they are applying to is also zero. We consider the case in which students have a degree of uncertainty about whether schools and colleges consider to be of high-ability, which can be arbitrarily small. That is, every student believes that there is an arbitrarily small probability $\epsilon > 0$ that she will be identified as high ability. Regardless of how small that value is, the probability of being accepted at both schools and colleges is therefore strictly positive. Moreover, since being accepted at some school or college is preferred over not being accepted at any, there is a strictly positive increase in the expected utility that comes from applying to each school and college. Since the cost of application is zero, what follows is the remark below. [5]

Remark 2. In equilibrium, students apply to all institutions.

The only question that remains is whether students would reject, in equilibrium, an offer from their most preferred school in order to increase their chances of being admitted to the most preferred college. We will focus on high-ability students.[6] The probability of being matched to college C_A, conditional on being admitted to schools S_1 and S_2 are as follows:

[5] Note that without this assumption, the equilibria described below will still be equilibrium, but there may exist additional "unreasonable" equilibria, involving subsets of students who are not high-ability, are not applying to schools and/or colleges, or are applying to them in a way that is inconsistent with their preferences.

[6] The focus on high-ability students is justified if we consider, as mentioned above, that students may not know whether they are high-ability students according to schools and colleges' criteria. Therefore, those who are not high-ability will still choose to apply to schools, even though they will never be chosen. Since they will not receive any offers from schools or colleges, they will not be able to strategically choose which offer to accept, justifying our focus.

$$P\left(C_A|S_1\right)=P\left(C_A|S_2\right)=\frac{\lambda_A^1}{H_1^*}=\frac{\lambda_A^2}{H_2^*}=\begin{cases}\frac{1}{1+\delta q} & \text{if } Q>H_A^*\text{, and}\\ \frac{Q}{H_1^*+H_2^*} & \text{otherwise.}\end{cases}$$

Let $\mathcal{H}^*=\frac{Q}{H_1^*+H_2^*}$. The probabilities of a student receiving an offer from college C_B, conditional on the school she attended, are:

$$P\left(C_B|S_1\right)=\frac{\lambda_B^1}{H_1^*}=\begin{cases}\frac{\delta q}{2+\delta q} & \text{if } Q>H_A^*\text{ and } Q>H_B^*,\\ \frac{1}{H_1^*}\left(\frac{H_1^*-\mathcal{H}^*}{\delta q+1-\mathcal{H}^*}\right) & \text{if } Q\leqslant H_A^*\text{ and } Q>H_B^*,\\ \frac{Q}{H_1^*+H_2^*-Q} & \text{otherwise,}\end{cases}$$

and

$$P\left(C_B|S_2\right)=\frac{\lambda_B^2}{H_2^*}=\begin{cases}\frac{\delta q}{2+\delta q} & \text{if } Q>H_A^*\text{ and } Q>H_B^*,\\ \frac{1}{H_2^*}\left(\frac{H_2^*-\mathcal{H}^*}{\delta q+1-\mathcal{H}^*}\right) & \text{if } Q\leqslant H_A^*\text{ and } Q>H_B^*,\\ \frac{Q}{H_1^*+H_2^*-Q} & \text{otherwise.}\end{cases}$$

Since students have zero mass, individuals have no impact on these values. Notice that $P\left(C_A|S_1\right)=P\left(C_A|S_2\right)$ and that unless C_A is binding but C_B is not binding (that is, if $Q>H_A^*$ and $Q>H_B^*$), $P\left(C_B|S_1\right)=P\left(C_B|S_2\right)$. That is, unless C_A is binding and C_B is not binding, there is no strategic value, regardless of the utility levels obtained by each school and college, in choosing which school to attend as a way to increase the chances of being matched to a more desired college. Moreover, even when there is potential gain from that choice, it consists of changing the likelihood of remaining unmatched instead of being matched to some college. With these, we can now proceed to the equilibria of the game induced on the students.

Theorem 1. *In the game induced on the students by the sequential screening game, the set of Perfect Bayesian Nash Equilibria is characterized by the following conditions:*

(i) If $Q>H_A^*$ and $Q>H_B^*$, there is a unique Perfect Bayesian Nash Equilibrium in which all students are truthful.

(ii) If $Q\leqslant H_A^*$ and $Q>H_B^*$, there is a value $v^*>0$ such that if $\frac{v_1^{1\succ 2}-v_2^{1\succ 2}}{v_B}\geqslant v^*$, there is a Perfect Bayesian Equilibrium in which all students are truthful. If $\frac{v_1^{1\succ 2}-v_2^{1\succ 2}}{v_B}\leqslant v^*$ and there is a value $\sigma^*>0$ that solves $H_A^*=Q$, there is a Perfect Bayesian Equilibrium in which a mass $\sigma-\sigma^*$ of students who prefer S_1 over S_2, accept offers from the latter whenever they receive one. If there is no such σ^*, there is a Perfect Bayesian Equilibrium in which every student, regardless of their

preferences, accepts offers from school S_2 whenever they are given.

(iii) If $Q \leqslant H_A^*$ and $Q \leqslant H_B^*$, there is a unique Perfect Bayesian Equilibrium in which all students are truthful.

Theorem 1 shows, therefore, that unless the parameters of the problem are such that college C_A is binding but C_B is not binding, there is no space for strategic behavior on the part of the students, with the objective of obtaining better assignments, even when students are almost indifferent between the two schools. The reason for that is that colleges, when deciding optimally how many students from each school they will screen, make choices that equate the marginal cost per high-ability student acquired in both schools. The cost of screening for those students, however, is proportional to its marginal scarcity in the pool, and as a result, colleges' screening choices consist of a fixed proportion of the number of them in each school, making the ratio "number of students screened from a school"/"number of high-ability students at a school" constant and equal across the two schools. College C_B, having a higher marginal cost per high-ability student acquired due to the effect that students' preferences have on it, face a shift in the cost curve, but the same optimization problem as C_A.

In contrast to the other two cases, when college C_A is binding and C_B is not, the proportion of the seats taken by students from the two schools is different between the two colleges, resulting in different probabilities of acceptance for the students from both schools. This happens because college C_B captures a higher proportion of the high-ability students from school S_2 than from S_1. Proposition 5 helps us to understand why this happens. It shows that, when college C_A is binding, a smaller capacity implies a stronger reduction in the number of students from S_2 than from S_1. That is, when college C_A's decision is binding, its decision will imply a greater reduction than otherwise in the screening of students from S_2. This creates a greater advantage for C_B to screen from S_2 than otherwise, leading to that asymmetry. Notice that when both colleges are binding, that reduction takes place in both colleges, and therefore the symmetry is restored.

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Appendix

Proofs of Propositions

Proposition 1

Item (i)

First, we want to show that $H_1^* < H_2^*$. Suppose, on the contrary, that

$$\frac{H_1^*}{H_2^*} = \left(\frac{1-\sigma}{\sigma}\right)\frac{(\kappa(\kappa+2) - A + \sigma(\sigma+1))}{(\kappa(\kappa+2) - A + (\sigma-3)\sigma+2)} \geqslant 1,$$

which implies $(1-2\sigma)(\kappa(\kappa+2)+(\sigma-1)\sigma - A) \geqslant 0$. Given $\sigma < \frac{1}{2}$, we have $\kappa(\kappa+2) + (\sigma-1)\sigma \geqslant A$. Using the definition of A, we have $8\kappa(1-\sigma)\sigma \leqslant 0$, which is a contradiction. Therefore, $H_1^* < H_2^*$.

Second, we want to show that $\lambda_1^* < \lambda_2^*$. Suppose not. Then

$$\frac{\lambda_1^*}{\lambda_2^*} = \frac{(\kappa-\sigma+1)(1-\sigma)(\kappa(\kappa+2) - A + \sigma(\sigma+1))}{(\kappa+\sigma)\sigma(\kappa(\kappa+2) - A + (\sigma-3)\sigma+2)} \geqslant 1,$$

which implies $(1-2\sigma)\left(\kappa^3 + 3\kappa^2 - \kappa\left(A - \sigma^2 + \sigma - 2\right) - A - \sigma^2 + \sigma\right) \geqslant 0$. Given that $\sigma < \frac{1}{2}$, we have $(1+\kappa)A \leqslant \kappa^3 + 3\kappa^2 + \kappa\left(2 - \sigma + \sigma^2\right) - \sigma(1-\sigma)$. Using the definition of A, we have $4\kappa(1-\sigma)\sigma(\kappa(1+\kappa) + \sigma(1-\sigma)) \leqslant 0$, which is a contradiction. Therefore, we have $\lambda_1^* < \lambda_2^*$.

Item (ii) First, note that $\frac{\partial A}{\partial \kappa} = \frac{4\kappa^3+12\kappa^2+4\kappa(2-\sigma(1-\sigma))+4\sigma(1-\sigma)}{2A}$. We separate the proof into two parts: (a) : $\frac{\partial \lambda_1^*}{\partial \kappa} < 0$; and (b) : $\frac{\partial \lambda_2^*}{\partial \kappa} < 0$.

Part (a): Recall that $\lambda_1^* = \frac{\alpha}{2\sigma}\frac{\kappa(\kappa+2)+\sigma(\sigma+1)-A}{(\kappa+\sigma)}$. We then have:

$$\frac{\partial \lambda_1^*}{\partial \kappa} = \frac{\alpha}{2\sigma}\frac{(\kappa+\sigma)\left(2\kappa+2-\frac{\partial A}{\partial \kappa}\right) - (\kappa(\kappa+2)+\sigma(\sigma+1) - A)}{(\kappa+\sigma)^2}.$$

Hence, it suffices to show that (i): $2\kappa + 2 - \frac{\partial A}{\partial \kappa} < 0$ and (ii):$\kappa(\kappa+2) + \sigma(\sigma+1) - A > 0$.

For (i), note that $2\kappa+2-\frac{\partial A}{\partial \kappa} = 2\kappa + 2 - \frac{4\kappa^3+12\kappa^2+4\kappa(2-\sigma(1-\sigma))+4\sigma(1-\sigma)}{2A}$ so that it suffices to show that $4(\kappa+1)A \geqslant 4\kappa^3 + 12\kappa^2 + 4\kappa(2-\sigma(1-\sigma)) + 4\sigma(1-\sigma)$. This is equivalent to $64\kappa\sigma(1-\sigma)\left(\kappa^2+\kappa-\sigma^2+\sigma\right) \geqslant 0$, which always holds because $-\sigma^2+\sigma \geqslant 0$, as $\sigma \in \left(0, \frac{1}{2}\right)$.

For (ii), note that $\kappa(\kappa+2) + \sigma(\sigma+1) > A \geqslant 0$ is equivalent to $(\kappa(\kappa+2) + \sigma(\sigma+1))^2 >$

$\kappa^4 + 4\kappa^3 + 2\kappa^2(2 - \sigma(1-\sigma)) + 4\kappa\sigma(1-\sigma) + (1-\sigma)^2\sigma^2$, which can be simplified as $4\sigma(\kappa+\sigma)^2 > 0$, which is always true.

Part (b) . Recall that $\lambda_2^* = \frac{\alpha}{2\sigma}\frac{\kappa(\kappa+2)+\sigma(\sigma-3)+2-A}{(\kappa+1-\sigma)}$, we have

$$\frac{\partial \lambda_2^*}{\partial \kappa} = \frac{\alpha}{2(1-\sigma)} \frac{(\kappa+1-\sigma)\left(2\kappa+2-\frac{\partial A}{\partial \kappa}\right) - (\kappa(\kappa+2)+\sigma(\sigma-3)+2-A)}{(\kappa+1-\sigma)^2}.$$

Note that we have proved in part (a) that $2\kappa + 2 - \frac{\partial A}{\partial \kappa} < 0$. Hence, $\frac{\partial \lambda_2^*}{\partial \kappa} < 0$ holds if $\kappa(\kappa+2) + \sigma(\sigma-3) + 2 > A$. This is equivalent to $(\kappa(\kappa+2) + \sigma(\sigma-3) + 2)^2 > \kappa^4 + 4\kappa^3 + 2\kappa^2(2 - \sigma(1-\sigma)) + 4\kappa\sigma(1-\sigma) + (1-\sigma)^2\sigma^2$ which can be simplified as $4(1-\sigma)(\kappa-\sigma+1)^2 > 0$, which is always true.

Item (iii) First note that $H_2^* = \frac{\alpha\left(\kappa^2-A-\sigma(1-\sigma)\right)(\kappa(\kappa+2)-A+\sigma(\sigma-3)+2)}{4(1-\sigma)(\sigma-1-\kappa)(\kappa+\sigma)} = \frac{\left(A+\sigma(1-\sigma)-\kappa^2\right)}{2(\kappa+\sigma)}\lambda_2^*$. We can show, through some algebra, that $\frac{\partial H_2^*}{\partial \kappa} < 0$ is equivalent to

$$AX + Y < 0$$

where

$$X = -2\kappa^5 - 6\kappa^4 - \kappa^3\left(6 + 4\sigma - 4\sigma^2\right) - \kappa^2\left(3 + 5\sigma - 6\sigma^2\right) - 2\kappa\sigma\left(4 - 9\sigma + 8\sigma^2 - 3\sigma^3\right) - (1-\sigma)^2\sigma, \text{ and}$$
$$Y = 2\kappa^7 + 10\kappa^6 + 2\kappa^5\left(9 + \sigma - \sigma^2\right) + \kappa^4\left(13 + 17\sigma - 16\sigma^2\right) + \kappa^3\left(2 + 32\sigma - 44\sigma^2 + 24\sigma^3 - 10\sigma^4\right)$$
$$+ 2\kappa\sigma\left(5 - 2\sigma - 6\sigma^2 + 3\sigma^3\right) + 2\kappa(1-\sigma)^2\sigma\left(1 + 5\sigma^2 - 3\sigma^3\right) - (1-\sigma)^3\sigma^2.$$

Clearly, $X < 0$. If $Y < 0$ then we are done. Suppose now that $Y > 0$. Then $AX + Y < 0$ is equivalent to $Y^2 - (AX)^2 < 0$, which in turn is equivalent to

$$\kappa^4 + 4\kappa^3(1-\sigma) + 2\kappa^2\left(2 + 2\sigma - \sigma^2\right) + 4\kappa\sigma\left(2 - 5\sigma + 3\sigma^2\right) + (1-\sigma)^2\sigma(2 - 3\sigma) > 0$$

which is always true.

Item (iv) First note that $H_1^* = \frac{\alpha\left(\kappa^2-A-\sigma(1-\sigma)\right)(\kappa(\kappa+2)-A+\sigma(\sigma+1))}{4\sigma(\sigma-1-\kappa)(\kappa+\sigma)} = \frac{\left(\kappa^2-A-\sigma(1-\sigma)\right)}{2(\sigma-1-\kappa)}\lambda_1^*$. We can show that $\frac{\partial H_1^*}{\partial \kappa} < 0$ is equivalent to

$$AX + Y < 0$$

where

$$X = -2\kappa^5 - 6\kappa^4 - \kappa^3\left(6+4\sigma-4\sigma^2\right) - \kappa^2\left(2+7\sigma-6\sigma^2\right) - 2\kappa\sigma\left(2-3\sigma+4\sigma^2-3\sigma^3\right) - (1-\sigma)\sigma^2, \text{ and}$$
$$Y = 2\kappa^7 + 10\kappa^6 + 2\kappa^5\left(9+\sigma-\sigma^2\right) + \kappa^4\left(14+15\sigma-16\sigma^2\right) + \kappa^3\left(4+24\sigma-32\sigma^2+16\sigma^3-10\sigma^4\right)$$
$$+ 2\kappa^2\sigma\left(5-2\sigma-6\sigma^2+3\sigma^3\right) + 2\kappa\sigma^2\left(3-4\sigma-3\sigma^2+7\sigma^3-3\sigma^4\right) + (1-\sigma)^2\sigma^3.$$

Clearly, $X < 0$ and $Y > 0$. Hence, $AX + Y < 0$ is equivalent to $Y^2 - (AX)^2 < 0$ which simplifies to $-\left(\kappa^4 + 4\kappa^3\sigma - 2\kappa^2\left(1-4\sigma+\sigma^2\right) - 4\kappa\sigma\left(1-4\sigma+3\sigma^2\right) + \sigma^2\left(-1+4\sigma-3\sigma^2\right)\right) < 0$. Hence, we have

$$\frac{\partial H_1^*}{\partial\kappa} < 0 \Leftrightarrow \kappa^4 + 4\kappa^3\sigma - 4\kappa\sigma\left(1-4\sigma+\sigma^2\right) - \left(\sigma^2+4\kappa\right)(1-\sigma)(1-3\sigma) > 0.$$

If $\sigma > \frac{1}{3}$ then we have $\frac{\partial H_1^*}{\partial\kappa} < 0$. Also, if $\kappa > 2$, then $\frac{\partial H_1^*}{\partial\kappa} < 0$. We now focus on $\sigma \in \left(0, \frac{1}{3}\right)$ and $\kappa \in (0,2)$. Let $g(\kappa) = \kappa^4 + 4\kappa^3\sigma - 4\kappa\sigma\left(1-4\sigma+\sigma^2\right) - \left(\sigma^2+4\kappa\right)(1-\sigma)(1-3\sigma)$. Then $g'(\kappa) = 4\kappa^3 + 12\kappa^3\sigma - 4\sigma\left(1-4\sigma+\sigma^2\right) - 4\kappa(1-\sigma)(1-3\sigma)$. Note that $g(0) < 0$ and $g(2) = 8 + 56\sigma + 23\sigma^2 - 50\sigma^3 - 3\sigma^4 > 0$. Hence, there will be at least one root between 0 and 2 for $g(\kappa) = 0$. However, $g'(\kappa) > 0$ for $\kappa \in (0,2)$. Therefore, by Rolle's theorem, there will only be one root.

Proposition 2

First, we have $H_1^* + H_2^* = \frac{\alpha}{4}\frac{(\kappa(\kappa+2)-A+3\sigma(1-\sigma))\left(A-\kappa^2+\sigma-\sigma^2\right)}{\sigma(1-\sigma)(\kappa+\sigma)(\kappa-\sigma+1)}$ and $H_A^2 = \frac{H_2^* Q}{H_1^*+H_2^*} = \frac{Q}{1+\frac{H_1^*}{H_2^*}} = \frac{Q}{1+\frac{1-\sigma}{\sigma}+\frac{1-\sigma}{\sigma}\frac{4\sigma-2}{\kappa(\kappa+2)-A+\sigma(\sigma-3)+2}}$. Note that

$$\frac{d}{d\kappa}\left(\kappa(\kappa+2) - A + \sigma(\sigma-3) + 2\right) =$$

$$= \frac{64\kappa\sigma(1-\sigma)(\kappa-\sigma+1)(\kappa+\sigma)}{2A\left((2\kappa+2)(2A) + \left(4\kappa^3+12\kappa^2+4\kappa\left(2-\sigma(1-\sigma)\right)+4\sigma(1-\sigma)\right)\right)} > 0.$$

Now, by proposition 1 , $H_1^* < H_2^*$. Therefore, $\frac{\partial H_A^2}{\partial Q} = \frac{H_2^*}{H_1^*+H_2^*} > \frac{\partial H_A^1}{\partial Q} = \frac{H_1^*}{H_1^*+H_2^*}$. Then we have

$$\frac{\partial H_A^2}{\partial \kappa} = -\frac{2Q(1-2\sigma)}{\left(2+\frac{4\sigma-2}{\kappa(\kappa+2)-A+\sigma(\sigma-3)+2}\right)^2}\frac{d}{d\kappa}(\kappa(\kappa+2)-A+\sigma(\sigma-3)+2) < 0.$$

Finally, given $H_A^1 + H_A^2 = Q$, we have $\frac{\partial\left(H_A^1+H_A^2\right)}{\partial \kappa} = 0$ and $\frac{\partial H_A^1}{\partial \kappa} > 0$.

Proposition 3 First, we have

$$\begin{aligned}
\frac{\partial \lambda_B^1}{\partial Q} &= \frac{-\left(\delta q + 1 - \frac{Q}{H_1^*+H_2^*}\right)\frac{1}{H_1^*+H_2^*} + \left(H_1^* - \frac{Q}{H_1^*+H_2^*}\right)\frac{1}{H_1^*+H_2^*}}{\left(\delta q + 1 - \frac{Q}{H_1^*+H_2^*}\right)^2} \\
&= \frac{-\left(\delta q + 1 - \frac{Q}{H_1^*+H_2^*}\right) + \left(H_1^* - \frac{Q}{H_1^*+H_2^*}\right)}{\left(\delta q + 1 - \frac{Q}{H_1^*+H_2^*}\right)^2}\frac{1}{H_1^*+H_2^*} < 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial \lambda_B^2}{\partial Q} &= \frac{-\left(\delta q + 1 - \frac{Q}{H_1^*+H_2^*}\right)\frac{1}{H_1^*+H_2^*} + \left(H_2^* - \frac{Q}{H_1^*+H_2^*}\right)\frac{1}{H_1^*+H_2^*}}{\left(\delta q + 1 - \frac{Q}{H_1^*+H_2^*}\right)^2} \\
&= \frac{-\left(\delta q + 1 - \frac{Q}{H_1^*+H_2^*}\right) + \left(H_2^* - \frac{Q}{H_1^*+H_2^*}\right)}{\left(\delta q + 1 - \frac{Q}{H_1^*+H_2^*}\right)^2}\frac{1}{H_1^*+H_2^*} < 0.
\end{aligned}$$

Proposition 4

Since college C_A is binding, $\frac{H_A^*}{Q} = 1$ and therefore $\frac{\partial}{\partial Q}\frac{H_A^*}{Q} = 0$. Now we are going to show $\frac{\partial}{\partial Q}\frac{H_B^*}{Q} < 0$. First, note that since $H_B^* > 0$, $\left(H_1^* + H_2^*\right)^2 \geqslant 2Q$. Second, given $\frac{H_1^*+H_2^*}{1+\delta q} > Q$, $\left(H_1^* + H_2^*\right)^3 \geqslant 2Q^2$. Now we have $\frac{H_B^*}{Q} = \frac{H_1^*+H_2^*-\frac{2Q}{H_1^*+H_2^*}}{\delta q+1-\frac{Q}{H_1^*+H_2^*}}\left(\frac{1}{Q} - \frac{1}{H_1^*+H_2^*}\right) =$

$\frac{\frac{H_1^*+H_2^*}{Q}-\frac{2}{H_1^*+H_2^*}-1+\frac{2Q}{\left(H_1^*+H_2^*\right)^2}}{\delta q+1-\frac{Q}{H_1^*+H_2^*}}$, and hence,

$$\frac{\partial}{\partial Q}\frac{H_B^*}{Q}=\frac{\left(\delta q+1-\frac{Q}{H_1^*+H_2^*}\right)\left(-\frac{H_1^*+H_2^*}{Q^2}+\frac{2}{\left(H_1^*+H_2^*\right)^2}\right)+\left(\frac{H_1^*+H_2^*}{Q}-\frac{2}{H_1^*+H_2^*}-1+\frac{2Q}{\left(H_1^*+H_2^*\right)^2}\right)\frac{1}{H_1^*+H_2^*}}{\left(\delta q+1-\frac{Q}{H_1^*+H_2^*}\right)^2}$$

$$=-\frac{\left(H_1^*+H_2^*\right)\left(H_1^*+H_2^*-Q\right)^2+q\delta\left(\left(H_1^*+H_2^*\right)^3-2Q^2\right)}{Q^2\left(H_1^*+H_2^*-Q+\left(H_1^*+H_2^*\right)q\delta\right)^2}<0$$

since $\left(H_1^*+H_2^*\right)^3-2Q^2\geqslant 0$.

Proposition 5

Item (i) We have

$$\frac{\partial\lambda_B^1}{\partial Q}=\frac{\left(H_1^*+H_2^*-Q\right)H_1^*+H_1^*Q}{\left(H_1^*+H_2^*-Q\right)^2}=\frac{\left(H_1^*+H_2^*\right)H_1^*}{\left(H_1^*+H_2^*-Q\right)^2},\text{ and}$$
$$\frac{\partial\lambda_B^2}{\partial Q}=\frac{\left(H_1^*+H_2^*-Q\right)H_2^*+H_2^*Q}{\left(H_1^*+H_2^*-Q\right)^2}=\frac{\left(H_1^*+H_2^*\right)H_2^*}{\left(H_1^*+H_2^*-Q\right)^2}.$$

Given that $H_2^*>H_1^*$,

$$\frac{\partial\lambda_B^2}{\partial Q}>\frac{\partial\lambda_B^1}{\partial Q}>0.$$

Item (ii) Follows directly from Proposition 2 (ii).

Proposition 6

Item (i) Note that $\lambda_1^*\lambda_2^*=\left(\frac{\kappa(\kappa+2)+\sigma(\sigma+1)-A}{2\sigma(\kappa+\sigma)}\alpha\frac{\kappa(\kappa+2)+\sigma(\sigma-3)+2-A}{2(1-\sigma)(\kappa+1-\sigma)}\alpha\right)$.

With some algebra, we can show that $\frac{\partial}{\partial\sigma}\lambda_1^*\lambda_2^*>0$ is equivalent to $9\kappa^4+3\kappa^5+\kappa^3\left(8+10\sigma-10\sigma^2\right)+\kappa^2\left(2+14\sigma-14\sigma^2\right)+\kappa\sigma\left(4-\sigma-6\sigma+3\sigma^3\right)+(1-\sigma)^2\sigma^2>0$. We can rewrite it as

$$AX+Y<0$$

where

$$X = 3\kappa^4 + 6\kappa^3 + 2\kappa^2\left(2 - \sigma + \sigma^2\right) + \kappa\left(1 - 2\sigma + 2\sigma^2\right) - (1-\sigma)^2\sigma^2, \text{ and}$$
$$Y = -3\kappa^6 - 12\kappa^5 - \kappa^4\left(18 - 5\sigma + 5\sigma^2\right) - 11\kappa^3 - \kappa^2\left(2 + 6\sigma - 5\sigma^2 - 2\sigma^3 + \sigma^4\right)$$
$$- \kappa\sigma\left(1 + 3\sigma - 8\sigma^2 + 4\sigma^3\right).$$

Note that Y is always negative. If $X \leqslant 0$, then we are done. Consider $X > 0$, then $AX + Y < 0$ is equivalent to

$$(AX)^2 - Y^2 < 0$$

which is equivalent to $9\kappa^4 + 3\kappa^5 + \kappa^3\left(8 + 10\sigma - 10\sigma^2\right) + \kappa^2\left(2 + 14\sigma - 14\sigma^2\right) + \kappa\sigma\left(4 - \sigma - 6\sigma + 3\sigma^3\right) + (1-\sigma)^2\sigma^2 > 0$ which is always true.

Item (ii) To show that $\frac{\partial H_1^*}{\partial \sigma} > 0$, we rely on the fact that $\frac{\partial H_2^*}{\partial \sigma} < 0$ and $\frac{\partial\left(H_1^* + H_2^*\right)}{\partial \sigma} < 0$, which are shown below. We can show that $\frac{\partial H_2^*}{\partial \sigma} < 0$ is equivalent to

$$AX + Y < 0$$

where

$$X = -\kappa^6 - 4\kappa^5 - \kappa^4\left(2 + 8\sigma - 5\sigma^2\right) + 2\kappa^3\left(1 - 7\sigma + 5\sigma^2\right)$$
$$+ \kappa^2\left(3 - 16\sigma + 24\sigma^2 - 16\sigma^3 + 5\sigma^4\right) + \kappa(1-\sigma)^2\left(1 - 4\sigma + 2\sigma^2\right) - (1-\sigma)^4\sigma^2, \text{ and}$$
$$Y = \kappa^8 + 6\kappa^7 + \kappa^6\left(10 + 7\sigma - 4\sigma^2\right) + \kappa^5\left(2 + 30\sigma - 20\sigma^2\right) + \kappa^4\left(-11 + 58\sigma - 62\sigma^2 + 29\sigma^3 - 10\sigma^4\right)$$
$$+ \kappa^3\left(-9 + 32\sigma - 25\sigma^2 + 2\sigma^4\right) - \kappa^2(1-\sigma)^2\left(2 + \sigma - 4\sigma^2 - 9\sigma^3 + 4\sigma^4\right) - (1-\sigma)^5\sigma^3.$$

Consider first the case $\kappa \geqslant 1$. We have $X < 0$. Then if $Y < 0$, the inequality ($AX + Y < 0$) is true. Now consider $Y > 0$. Then $AX + Y < 0$ is equivalent to

$$Y^2 - (AX)^2 < 0$$

which is always true.

Now consider $\kappa \in (0, 1)$. First, $\sigma > \frac{1}{4}(3 + 4\kappa) - \frac{1}{4}\sqrt{1 + 16\kappa + 32\kappa^2}$ implies $X < 0$. Then if $Y < 0$, the inequality ($AX + Y < 0$) is true. Now consider $Y > 0$. Then $AX + Y < 0$ is equivalent to

$$Y^2 - (AX)^2 < 0.$$

If $\sigma < \frac{1}{4}(3+4\kappa) - \frac{1}{4}\sqrt{1+16\kappa+32\kappa^2}$, $Y<0$. Then if $X<0$, the inequality ($AX+Y<0$) is true. Now consider $X>0$. Then $AX+Y<0$ is equivalent to

$$(AX)^2 - Y^2 < 0,$$

which can be shown to be true.

Item (iii) Note that

$$\frac{\partial\left(H_1^* + H_2^*\right)}{\partial\sigma} = \frac{\alpha\kappa\left(1-2\sigma\right)\left(-\kappa^3 - 4\kappa^2 + \kappa\left(A+\sigma-\sigma^4-4\right) + 2(A-(1-\sigma)\sigma)\right)}{2\left(1-\sigma\right)^2\sigma^2 A}.$$

Since the denominator is positive, $\frac{\partial\left(H_1^*+H_2^*\right)}{\partial\sigma} > 0$ is equivalent to $-\kappa^3 - 4\kappa^2 + \kappa\left(A+\sigma-\sigma^4-4\right) + 2(A-(1-\sigma)\sigma) > 0$. We can rewrite the above as

$$AX + Y > 0$$

where

$$X = -\left(2+\kappa\right), \text{ and}$$
$$Y = \kappa^3 + 4\kappa^2 + \kappa\left(4-\sigma+\sigma^2\right) + 2\left(1-\sigma\right).$$

Since $X<0$, and $Y>0$, then $AX+Y>0$.

Proposition 7

We can show that

$$\frac{\partial\mathcal{H}^{1,2}}{\partial\sigma} = \lambda_1^* - \lambda_2^* + \sigma\frac{\partial\lambda_1^*}{\partial\sigma} + \left(1-\sigma\right)\frac{\partial\lambda_2^*}{\partial\sigma} < 0$$

is equivalent to

$$8\kappa\left(\kappa+\kappa^2+\sigma-\sigma^2\right)^2 > 0,$$

which is always true. Since $\frac{\partial \mathcal{H}^{1,2}}{\partial \sigma} < 0$ and $\frac{\partial\left(H_1^*+H_2^*\right)}{\partial \sigma} > 0$, we have

$$\frac{\partial}{\partial \sigma} \frac{\mathcal{H}^{1,2}}{H_1^* + H_2^*} = \frac{\left(H_1^* + H_2^*\right) \frac{\partial \mathcal{H}^{1,2}}{\partial \sigma} - \mathcal{H}^{1,2} \frac{\partial\left(H_1^*+H_2^*\right)}{\partial \sigma}}{\left(H_1^* + H_2^*\right)^2} < 0.$$

Proposition 8

By Proposition 6, $\frac{\partial\left(H_1^*+H_2^*\right)}{\partial \sigma} > 0$. When no college is binding, $H_A^* = \frac{H_1^*+H_2^*}{1+\delta q}$ and $H_B^* = \frac{\delta q\left(H_1^*+H_2^*\right)}{(1+\delta q)(2+\delta q)}$, therefore $\frac{\partial H_A^*}{\partial \sigma} > 0$ and $\frac{\partial H_B^*}{\partial \sigma} > 0$. When C_A is binding, $H_B^* = \left(\frac{H_1^*+H_2^*-\frac{2Q}{H_{12}^*}}{\delta q+1-\frac{Q}{H_1^*+H_2^*}}\right)\left(1 - \frac{Q}{H_1^*+H_2^*}\right)$. Since $\frac{\partial H_B^*}{\partial\left(H_1^*+H_2^*\right)} > 0$, in that case $\frac{\partial H_B^*}{\partial \sigma} > 0$.

Proposition 9

Consider first the case in which none of the colleges are binding. We have $H_A^* + H_B^* = \frac{H_1^*+H_2^*}{1+q\delta} + \frac{\delta q\left(H_1^*+H_2^*\right)}{(1+\delta q)(2+\delta q)} = \left(H_1^* + H_2^*\right)\left(\frac{2}{q\delta+2}\right)$ and $\frac{\mathcal{H}^{A,B}}{H_A^*+H_B^*} = \frac{\mathcal{H}^{1,2}}{H_1^*+H_2^*}\frac{2+\delta q}{2(1+\delta q)}$. Hence, we have

$$\frac{\partial \mathcal{H}^{A,B}}{\partial \sigma} = \frac{1}{1+\delta q}\frac{\partial \mathcal{H}^{1,2}}{\partial \sigma} < 0,$$

and

$$\frac{\partial}{\partial \sigma}\frac{\mathcal{H}^{A,B}}{H_A^* + H_B^*} = \frac{2+\delta q}{2\left(1+\delta q\right)}\frac{\partial}{\partial \sigma}\frac{\mathcal{H}^{1,2}}{H_1^* + H_2^*} < 0.$$

Next, we consider the case in which only C_A is binding. We have $\mathcal{H}^{A,B} = \frac{\mathcal{H}^{1,2}}{H_1^*+H_2^*}Q$ and $\frac{\mathcal{H}^{A,B}}{H_A^*+H_B^*} = \frac{\frac{\mathcal{H}^{1,2}}{H_1^*+H_2^*}Q}{\left(H_1^*+H_2^*\right)\left(\frac{2}{q\delta+2}\right)} = \frac{\mathcal{H}^{1,2}}{\left(H_1^*+H_2^*\right)^2}\frac{Q(2+\delta q)}{2}$. Hence, we have

$$\frac{\partial}{\partial \sigma}\mathcal{H}^{A,B} = Q\frac{\partial}{\partial \sigma}\frac{\mathcal{H}^{1,2}}{H_1^* + H_2^*} < 0$$

and

$$\begin{aligned}\frac{\partial}{\partial \sigma}\frac{\mathcal{H}^{A,B}}{H_A^* + H_B^*} &= \frac{Q\left(2+\delta q\right)}{2}\frac{\partial}{\partial \sigma}\frac{\mathcal{H}^{1,2}}{\left(H_1^* + H_2^*\right)^2} \\ &= \frac{Q\left(2+\delta q\right)}{2}\frac{\left(H_1^* + H_2^*\right)^2 \frac{\partial \mathcal{H}^{1,2}}{\partial \sigma} - 2\mathcal{H}^{1,2}\left(H_1^* + H_2^*\right)\frac{\partial}{\partial \sigma}\left(H_1^* + H_2^*\right)}{\left(H_1^* + H_2^*\right)^4} < 0\end{aligned}$$

because $\frac{\partial \mathcal{H}^{1,2}}{\partial \sigma} < 0$ and $\frac{\partial \left(H_1^* + H_2^*\right)}{\partial \sigma} > 0$.

Theorem 1

The expected utility that a student who, if she receives offers from both schools, accepts the offer from S_i, is the following (noting that no student would accept an offer from C_B if she also has one from C_A):

$$v_i + P\left(C_A|S_i\right) v_A + \left(1 - P\left(C_A|S_i\right)\right) P\left(C_B|S_i\right) v_B$$

Therefore, a student will accept the offer from school S_i over one from S_j when:

$$v_i + P\left(C_A|S_i\right) v_A + \left(1 - P\left(C_A|S_i\right)\right) P\left(C_B|S_i\right) v_B \geqslant v_j + P\left(C_A|S_j\right) v_A + \left(1 - P\left(C_A|S_j\right)\right) P\left(C_B|S_j\right) v_B$$

Since $P\left(C_A|S_1\right) = P\left(C_A|S_2\right)$, when denoting that value simply by $P\left(C_A\right)$, this is equivalent to:

$$v_i - v_j \geqslant \left(1 - P\left(C_A\right)\right)\left(P\left(C_B|S_j\right) - P\left(C_B|S_i\right)\right) v_B.$$

One can easily check that, since $H_2^* > H_1^*$, $P\left(C_B|S_2\right) \geqslant P\left(C_B|S_1\right)$ in every case. Since $v_2^{2\succ 1} > v_1^{2\succ 1}$, therefore, students who prefer school S_2 over S_1 would never accept school S_1 over S_2. We also know that, except when $Q \leqslant H_A^*$ and $Q > H_B^*$, $P\left(C_B|S_1\right) = P\left(C_B|S_2\right)$. Therefore, in those cases the students who prefer school S_1 would strictly prefer to behave truthfully when choosing a school, so no equilibrium other than the truthful is possible.

What is left is to analyze the choices made by the students who prefer school S_1 in the case in which $Q \leqslant H_A^*$ and $Q > H_B^*$. A student with that preference will choose to accept an offer from S_2, when facing offers from both schools, when:

$$v_2^{1\succ 2} - v_1^{1\succ 2} \geqslant \left(1 - P\left(C_A\right)\right)\left(P\left(C_B|S_1\right) - P\left(C_B|S_2\right)\right) v_B.$$

Or, equivalently:

$$\frac{v_1^{1\succ 2} - v_2^{1\succ 2}}{v_B} \leqslant \left(1 - P\left(C_A\right)\right)\left(P\left(C_B|S_2\right) - P\left(C_B|S_1\right)\right).$$

Therefore, unless the condition above holds, truth-telling by all students is a PBE. The last question we should ask is whether there is a PBE when that is not the case. That is, is there an equilibrium in which one or more students deviate from truthful behavior? In order to check that, we need to evaluate how the right-hand side of the expression above changes when students behave in a non-truthful manner. Since the only students who could behave in such a way are those who prefer school S_1 over S_2, situations in which some of them do so are equivalent to lower values of σ. This is so because these students will behave as if they actually preferred school S_2, and because the solution concept used, – the Perfect Bayesian Equilibrium – requires that schools update their beliefs over the number of students who act in such way. In other words, checking whether equilibria in which students are non-truthful consists of checking how the deviating condition above changes when the value of σ decreases.

From Proposition 6 we know that $\frac{\partial(H_1^*+H_2^*)}{\partial\sigma} > 0$, so a reduction in σ will imply an increase in $P(C_A)$. Regarding $P(C_B|S_1)$ and $P(C_B|S_2)$, we need two auxiliary results.

Claim 1. When only college C_A is binding, (i) $\frac{\partial}{\partial\sigma}\frac{\lambda_B^1}{H_1^*} > 0$ and (ii) $\frac{\partial}{\partial\sigma}\frac{\lambda_B^2}{H_2^*} < 0$.

Proof. (i) Given $\lambda_B^1 = \frac{H_1^*-\frac{Q}{H_1^*+H_2^*}}{\delta q+1-\frac{Q}{H_1^*+H_2^*}}$, we have $\frac{\lambda_B^1}{H_1^*} = \frac{1-\frac{Q}{H_1^*+H_2^*}\frac{1}{H_1^*}}{\delta q+1-\frac{Q}{H_1^*+H_2^*}}$. Hence,

$$\begin{aligned}\left(\delta q+1-\frac{Q}{H_1^*+H_2^*}\right)^2\frac{d}{d\sigma}\frac{\lambda_B^1}{H_1^*} &= -\left(\delta q+1-\frac{Q}{H_1^*+H_2^*}\right)\frac{d}{d\sigma}\left(\frac{Q}{H_1^*+H_2^*}\frac{1}{H_1^*}\right)\\ &\quad+\left(1-\frac{Q}{H_1^*+H_2^*}\frac{1}{H_1^*}\right)\frac{d}{d\sigma}\left(\frac{Q}{H_1^*+H_2^*}\right)\\ &\geqslant\left(1-\frac{\delta q+1}{H_1^*}\right)\frac{d}{d\sigma}\left(\frac{Q}{H_1^*+H_2^*}\right)>0\end{aligned}$$

where the first inequality follows from $\frac{d}{d\sigma}\left(\frac{1}{H_1^*}\right) < 0$, and the second inequality follows from $\frac{d}{d\sigma}\left(H_1^*+H_2^*\right) < 0$ (Proposition 6) and $1-\frac{\delta q+1}{H_1^*} < 0$ (because $0 \leqslant \lambda_B^1 \leqslant 1$).

(ii) Now, given that $\lambda_B^2 = \frac{H_2^*-\frac{Q}{H_1^*+H_2^*}}{\delta q+1-\frac{Q}{H_1^*+H_2^*}}$, we have $\frac{\lambda_B^2}{H_2^*} = \frac{1-\frac{Q}{H_1^*+H_2^*}\frac{1}{H_2^*}}{\delta q+1-\frac{Q}{H_1^*+H_2^*}}$. Hence, we have

$$\begin{aligned}\frac{d}{d\sigma}\frac{\lambda_B^1}{H_1^*} &=\frac{d}{d\sigma}\frac{1-\frac{Q}{H_1^*+H_2^*}\frac{1}{H_1^*}}{\delta q+1-\frac{Q}{H_1^*+H_2^*}}\\ &=\frac{1}{\left(\delta q+1-\frac{Q}{H_1^*+H_2^*}\right)^2}\left(\begin{array}{c}\left(\delta q+1-\frac{Q}{H_1^*+H_2^*}\right)\frac{Q}{\left(H_1^*+H_2^*\right)^2\left(H_1^*\right)^2}\frac{d}{d\sigma}\left(H_1^*\left(H_1^*+H_2^*\right)\right)\\ -\left(1-\frac{Q}{H_1^*+H_2^*}\frac{1}{H_1^*}\right)\frac{Q}{\left(H_1^*+H_2^*\right)^2}\frac{d}{d\sigma}\left(H_1^*+H_2^*\right)\end{array}\right)<0\end{aligned}$$

if $\frac{d}{d\sigma}\left(H_1^*\left(H_1^*+H_2^*\right)\right) < 0$, because $\frac{d}{d\sigma}\left(H_1^*+H_2^*\right) > 0$, $\delta q+1-\frac{Q}{H_1^*+H_2^*} > 0$ and $1-\frac{Q}{H_1^*+H_2^*}\frac{1}{H_1^*} > 0$. Therefore, we only need to show $\frac{d}{d\sigma}\left(H_1^*\left(H_1^*+H_2^*\right)\right) < 0$.

First, we have $\frac{d}{d\sigma}\left(H_1^*\left(H_1^*+H_2^*\right)\right) = \frac{XA^2+YA+Z}{D}\alpha^2$, where

$$
\begin{aligned}
X =& \kappa^6(1-3\sigma)+5\kappa^5(1-3\sigma)+\kappa^4\left(9-24\sigma-6\sigma^2+5\sigma^3\right)+\kappa^3\left(7-11\sigma-26\sigma^2+22\sigma^3\right)\\
&+\kappa^2\left(2+5\sigma-39\sigma^2+48\sigma^3-23\sigma^4+7\sigma^5\right)+\kappa(1-\sigma)^2\sigma\left(4-9\sigma+\sigma^2\right)-(1-\sigma)^4\sigma^3,\\
Y =& 2(-\kappa^8(1-3\sigma)-6\kappa^7(1-3\sigma)-2\kappa^6\left(7-20\sigma-\sigma^2+\sigma^3\right)-2\kappa^5\left(8-19\sigma-10\sigma^2+9\sigma^3\right)\\
&+\kappa^4\left(-9+9\sigma+56\sigma^2-70\sigma^3+34\sigma^4-12\sigma^5\right)-2\kappa^3\left(1+4\sigma-27\sigma^2+38\sigma^3-23\sigma^4+7\sigma^5\right)\\
&-2\kappa^2(1-\sigma)^2\sigma\left(2-5\sigma+5\sigma^2-7\sigma^3+3\sigma^4\right)-2\kappa(1-\sigma)^4\sigma^3-(1-\sigma)^5\sigma^4,\\
Z =& \kappa^{10}(1-3\sigma)+\kappa^9(7-21\sigma)+\kappa^8\left(19-56\sigma+2\sigma^2-\sigma^3\right)+\kappa^7\left(25-67\sigma-6\sigma^2+8\sigma^3\right)\\
&+\kappa^6\left(16-33\sigma-32\sigma^2+57\sigma^3-38\sigma^4+15\sigma^5\right)+\kappa^5\left(4-6\sigma-24\sigma^2+88\sigma^3-96\sigma^4+34\sigma^5\right)\\
&-\kappa^4\sigma\left(6+3\sigma-117\sigma^2+221\sigma^3-167\sigma^4+72\sigma^5-18\sigma^6\right)\\
&+\kappa^3(1-\sigma)^2\alpha\left(-4-19\sigma+77\sigma^2-26\sigma^3+8\sigma^4\right)\\
&+\kappa^2(1-\sigma)^3\sigma\left(-10+13\sigma+12\sigma^2+12\sigma^3-5\sigma^4\right)\\
&+\kappa(1-\sigma)^4\sigma^3\left(-4+5\sigma+3\sigma^2\right)-(1-\sigma)^6\sigma^5, \text{ and}\\
D =& 4(1+\kappa-\sigma)^2(1-\sigma)^3\sigma^2(\kappa+\sigma)^2 A.
\end{aligned}
$$

Since $D > 0$, it suffices to show $XA^2+YA+Z = X\left(A+\frac{Y}{2X}-\frac{\sqrt{Y^2-4XZ}}{2X}\right)\left(A+\frac{Y}{2X}+\frac{\sqrt{Y^2-4XZ}}{2X}\right) > 0$.

Note that $Y^2-4XZ = \Delta^2$ where

$$
\Delta = 2\kappa\left(\begin{array}{c}
\kappa^6(1-3\sigma)+\kappa^5(1-3\sigma)+\kappa^4\left(9-22\sigma-14\sigma^2+11\sigma^3\right)\\
+\kappa^3\left(7-5\sigma-50\sigma^2+40\sigma^3\right)+\kappa^2\left(2+11\sigma-55\sigma^2+38\sigma^3+13\sigma^4-9\sigma^5\right)\\
+3\kappa(1-\sigma)^2\sigma\left(2-\sigma-7\sigma^2\right)+(1-\sigma)^3\sigma^2\left(-4+7\sigma+\sigma^2\right)
\end{array}\right).
$$

There are two cases depending on the sign of Δ. Case (a): suppose $\Delta \geqslant 0$. Then we have

$$\begin{aligned}XA^2+YA+Z=&X\left(A+\frac{Y}{2X}-\frac{\Delta}{2X}\right)\left(A+\frac{Y}{2X}+\frac{\Delta}{2X}\right)\\=&\left(XA+\frac{Y-\Delta}{2}\right)\left(A-\left(\kappa^2+(1-\sigma)\sigma\right)\right)\end{aligned}$$

Note that $A-\left(\kappa^2+(1-\sigma)\sigma\right)>0$ is equivalent to $4\kappa\left(\kappa+\kappa^2+\sigma-\sigma^2\right)>0$ which is always true. Hence, we only need to show that

$$XA+\frac{Y-\Delta}{2}<0,$$

which is true because

$$\begin{aligned}&(\Delta-Y)^2-(2XA)^2\\&=32\kappa(1+\kappa-\sigma)^2(1-\sigma)^3\sigma^2\begin{pmatrix}2\kappa^7(1-3\sigma)+2\kappa^6\left(3-5\sigma-12\sigma^2\right)\\+\kappa^5\left(5+17\sigma-90\sigma^2+10\sigma^3\right)\\+\kappa^4\left(1+27\sigma-48\sigma^2-102\sigma^3+76\sigma^4\right)\\+\kappa^3\sigma\left(7+31\sigma-154\sigma^2+90\sigma^3+14\sigma^4\right)\\+5\kappa^2\sigma^2\left(3-5\sigma-12\sigma^2+22\sigma^3-8\sigma^4\right)\\+\kappa(1-\sigma)^2\sigma^3\left(11-22\sigma-2\sigma^2\right)\\+\sigma^32(1+\sigma)^3\sigma^4(1-2\sigma)\end{pmatrix}>0.\end{aligned}$$

Case (b): When $\Delta<0$, similar calculation shows that $XA^2+YA+Z<0$. Therefore, we can conclude that $\frac{d}{d\sigma}\left(H_1^*\left(H_1^*+H_2^*\right)\right)=\frac{XA^2+YA+Z}{D}\alpha^2<0$ and thus $\frac{d}{d\sigma}\frac{\lambda_B^1}{H_1^*}<0$. □

We conclude, therefore, that as more students deviate from truthful behavior and behave as if they preferred S_2 over S_1, $P(C_B|S_1)$ decreases and $P(C_B|S_2)$ increases. That is, the incentive that students have for deviating is reinforced. This would indicate that when the condition on utilities above holds there will be a unique PBE in which all students behave as if they prefer school S_2 over S_1. There is, however, one last possibility. Since $\frac{\partial\left(H_1^*+H_2^*\right)}{\partial\sigma}>0$, a reduction in σ results in a reduction in both H_A^* and H_B^*. There is, therefore, the possibility that college C_A will stop binding before σ reaches zero. If that happens, $P(C_B|S_2)=P(C_B|S_1)$ and therefore students will stop having an incentive to deviate. If C_A is still binding when $\sigma=0$, however, that will never be the case. To see how many high-ability students would be obtained by C_A in that case, we must take the limit of H_A^*.

$$\lim_{\sigma \to 0} H_A^* = \frac{2\alpha}{(1+\delta q)(2+\kappa)}.$$

Hence, if $Q \leqslant \frac{2\alpha}{(1+\delta q)(2+\kappa)}$, C_A will bind regardless of the value of σ. The last case to consider is when there are values of σ such that $H_A^* \leqslant Q$. That is, if there is value $0 < \sigma^* < \frac{1}{2}$ that solves $H_A^* = Q$, then the unique PBE, in this case, will be one in which a mass $\sigma - \sigma^*$ of students deviate from truthful behavior.

Therefore, if we denote by v^* the value of $(1 - P(C_A))(P(C_B|S_2) - P(C_B|S_1))$ when only college C_A is binding, we have proven our theorem:

$$v^* = \left(1 - \frac{Q}{H_1^* + H_2^*}\right)\left(\frac{1}{H_2^*}\left(\frac{H_2^* - \mathcal{H}^*}{\delta q + 1 - \mathcal{H}^*}\right) - \frac{1}{H_1^*}\left(\frac{H_1^* - \mathcal{H}^*}{\delta q + 1 - \mathcal{H}^*}\right)\right).$$

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